Introduction to Power Series

So far most of the series we have looked at has had the form $\sum_{n=1}^{\infty} a_n$ where the a_n is some sequence of constants. However, there is no reason why a_n should *have* to consist of constants. Recall the precise definition of a sequence: a sequence is a function whose domain is the set of positive integers and whose range is the real numbers. But why restrict the range of these functions to the real numbers? We could define sequences to include lists of polynomials or other functions of x that somehow depend on the index n. Once we start adding up these sequences of functions...things get interesting.

Let $a_n = \{x^n\}_{n=0}^{n=\infty}$ be the sequence of functions that sends each natural number to its corresponding power of x. So $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} x^n$

1) Write out the partial sum $\sum_{n=0}^{6} x^n$ here:

This series looks a lot like the geometric series we've studied. All the rules we've had before for manipulating series still apply (reindexing, sums, convergence tests). Yet, the questions we've asked about series in the past suddenly grow much more mysterious:

What does it really mean for $\sum_{n=0}^{\infty} x^n$ to converge? What could it possibly converge *to*?

Let's investigate these questions for this particular series:

2) Try plugging $x = 2$ into $\sum_{n=0}^{\infty} x^n$, and evaluate the sum or show it diverges:

3) Try plugging $x = -1/2$ into $\sum_{n=0}^{\infty} x^n$, and evaluate the sum or show it diverges:

It appears that this series converges only for certain values of x. We say x_0 is in the **interval of convergence** of the series if plugging in $x = x_0$ into to series yields a convergent numerical series. The series converges **to** a function of x within this interval. This function takes values x_0 and spits out another value: $\sum_{n=0}^{\infty}(x_0)^n$

4) Find the interval of convergence for $\sum_{n=0}^{\infty} x^n$.

5) The series converges to a function f in it's interval of convergence. What is this $f(x)$?

6) Find $\sum_{n=0}^{6} x^n$ when $x = 3$ as a decimal and compare it to $f(3)$

A **power series about x=a** is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ where a is a constant and c_n is a sequence of constants. The series we studied above is a power series about $x = 0$ where $c_n = 1$ for all n.

Let's consider a slightly more complicated power series where $a = 0$ and $c_n = n!$

6) Write out $\sum_{n=0}^{6} \frac{x^n}{n!}$ here. Note: $0! = 1$

7) Use a convergence test to find the interval of convergence of this series.

If x is in the interval of convergence, this series converges to some mystery function $f(x)$.

We know that derivatives respect sums, i.e. $\frac{d(h(x)+g(x))}{dx} = \frac{dh(x)}{dx} + \frac{dg(x)}{dx}$

There is no reason that derivatives should not respect *infinite sums.*

8) a) Take the derivative of your answer to (6) term by term below.

b) Express the derivative of the nth partial sum of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ using sigma notation.

c) The limit of the new sequence of partial sums you found in (b) will be a power series *converging to f'(x), the derivative of our mystery function.* Write this series below:

9) Go through the same steps, integrating $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ term by term to get a new power series converging to F, an antiderivative of the mystery function f.
Use the initial condition $F(0) = 1$ to get the best antiderivative.

10) Do you notice anything unusual about your answers to 8 and 9?
What is the mystery function f?

It looks like summing up a bunch of polynomials can get you some interesting functions. Let's try to go in the reverse direction: start with a given function $f(x)$, and try to approximate it using a sum of polynomials. We will assume $f(x)$ is continuously differentiable and well defined at zero.

There is exactly one constant function who value agrees with $f(x)$ at $x = 0$. It is $y = f(0)$.

Similarly, there is one polynomial of degree 1 whose value *and* first derivative agree with $f(x)$ at $x = 0$, the *linearization* of f. It is written $y = f(0) + f'(0)\,x$. We've used this polynomial to approximate values of f before, as the first step in the Euler method.

11) Starting with the linearization of f, find a polynomial of degree 2 whose value, first derivative, *and* second derivative agree with f at zero. (Find the coefficient of x^2).

12) Find a polynomial of degree 3 whose value, first derivative, 2nd derivative, *and* 3rd derivative agree with f at zero. Be sure you find the coefficient of x^3 carefully.

13) Find a polynomial of degree 4 whose value, first derivative, 2nd derivative, 3rd derivative, *and* 4th derivative agree with f at zero. At this point, you should double check your answers by taking all of this polynomial's derivatives and evaluating them at $x = 0$.

14) Let P_n be the polynomial of degree n that agrees with the first n derivatives of f. We found P_0, P_1, P_2, P_3, and P_4 above. Using sigma notation, write a formula for P_n here. Your answer should include $f^{(i)}$, the i^{th} derivative of f. Stuck? Try finding P_5 first.

15) Write an infinite series that expresses $\lim_{n\to\infty} P_n$ when $f(x) = e^x$. What is the interval of convergence of this series, and what does it converge to in that interval?

16*) In general, what do you think $\lim_{n\to\infty} P_n$ will be when it converges?